Implicit Differentiation

Math 130 - Essentials of Calculus

23 March 2021

Tangent Line to a Circle

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$$x^2 + y^2 = 1.$$

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1. We could solve the equation for y and take the derivative of that. When solving for y, we get $y = \pm\sqrt{1 - x^2}$, and we pick either the $+$ or $-$ part based on whether we are on the top half or bottom half of the circle, respectively. While this isn't *terribly* complicated here, for curves more complicated than a circle like this, it could be hard to solve for y.

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2. Use the technique of implicit differentiation and not have to solve explicitly for y.

Implicit Differentiation

Through the equation

$$x^2 + y^2 = 1$$

we say that y *is implicitly defined as a function of* x. This means that the value of y can be (possibly not uniquely) determined from this equation by plugging in a value for x. We call x the *independent variable* and y the *dependent variable*. In some sense, $y = f(x)$, but we cannot actually solve for the $f(x)$. Using this, we can still compute the derivative $\frac{dy}{dx} = f(x)$.

Example

Using implicit differentiation, find $\frac{dy}{dx}$ *for the unit circle* $x^2 + y^2 = 1$ *and find an equation for the tangent line passing though* $(1, \sqrt{3})$*.*

Implicit Differentiation

While we *could* have explicitly solved for y in the previous example, the following examples are equations in which it would be infeasible to do so:

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Find $\frac{dy}{dx}$ *for the following equations*

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Find $\frac{dy}{dx}$ *for the following equations*

1. $x^3 + y^3 = 6xy$
2. $x^2 - y^2 = 9$
3. $x^3 + x^2y + 4y^2$

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The *inverse* of an exponential function is called a *logarithmic function*.

Definition

Let $b > 0$ be a real number with $b \neq 1$. The logarithm with base b is the function

$$f(x) = \log_b x$$

which satisfies the property

$$\log_b x = y \iff b^y = x.$$

When the base of the logarithm is the natural number e, we write $\ln x$ *instead and call it the natural logarithm.*

Example - Computing Logarithms - No Calculators!

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Find the exact value of the following

1. $\log_2 64$

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4. $\ln e^3$

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5. $e^{\ln 4}$

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7. $\ln 1$

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8. $\ln e^x$

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9. $e^{\ln x}$

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The only function type that we've covered and not yet taken the derivative of is a logarithmic function. We can accomplish this derivative by using implicit differentiation and rewriting $y = \ln x$.

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In this case, we'd really like an expression for the derivative that has only x in it, and since we know $e^y = x$, just plug that in to get

$$\frac{dy}{dx} = \frac{1}{x}.$$

Examples

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Find the derivative of the given functions

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1. $y = 3x - 2\ln x$
2. $f(x) = \ln x^2$
3. $y = \sqrt[5]{\ln x}$
4. $y = x\ln(1 + e^x)$